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A FINITE ELEMENT METHOD FOR THE SOLUTION OF ONE-DIMENSIONAL PHASE CHANGE PROBLEMS

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NOMENCLATURE

- c, specific heat;
- k, thermal conductivity;
- L, latent heat;
- s, temporal position of the phase front;
- t, time;
- T, temperature;
- T_{Ph}, phase-change temperature;
- x, length;
- α, thermal diffusivity;
- θ, homogenised temperature;
- ρ, density.

INTRODUCTION

A CHARACTERISTIC of heat transfer problems dealing with phase change is the existence of an interface separating the phases. On this interface, the latent heat associated with the phase change is absorbed or liberated and, as a consequence of heat transfer processes in the two phases, the location of the interface is continuously changing in time. Extensive reviews on methods currently in use for the solution of heat transfer problems with phase change can be found in references [1, 2]. Several numerical methods using the finite element formulation have been reported in the technical literature [3–6]. The finite element methods mentioned in references [4–6] employ the enthalpy method of formulation and cannot cope with the case in which the phase change occurs at a specific temperature. The purpose of this note is to present a simple 1-dim. procedure using finite elements for solving heat transfer problems in which the change of phase occurs at a specific temperature.

FORMULATION OF THE PROBLEM

For the sake of simplicity it is assumed that the medium has homogeneous, isotropic thermal properties in each of its thermodynamic states, that the change of phase occurs at a specific temperature, and that boundary conditions of the first kind are imposed on the outer surface. This situation is governed by the parabolic equation

$$\rho \cdot c_m \frac{\partial \theta_m}{\partial t} = k_m \cdot \frac{\partial^2 \theta_m}{\partial x^2}, \tag{1}$$

where $m = 1, 2$ for the different thermodynamic states and

$$\theta_m = T_m - T_{Ph}. \tag{2}$$

The initial conditions in the region of interest are

$$\theta_m(x, 0) = f_1(x), \tag{3}$$

and the boundary conditions on the outer surface Γ are

$$\theta_m(\Gamma, t) = f_2(t). \tag{4}$$

A boundary condition is also imposed on the phase interface:

$$\theta_m(s(t), t) = 0. \tag{5}$$

The energy balance on the phase interface is

$$k_1 \frac{\partial \theta_1}{\partial x} - k_2 \frac{\partial \theta_2}{\partial x} = \rho \cdot L \cdot \frac{\mathrm{d}s(t)}{\mathrm{d}t} \quad \text{at } x = s(t). \tag{6}$$

The 1-dim. space and the attendant temperature field can be divided into isoparametric elements with variable numbers of nodes such that for an element with N nodes,

$$x = \sum_{i=1}^{N} h_i x_i \tag{7}$$

and

$$\theta = \sum_{i=1}^{N} h_i \theta_i,$$

where x_i are the finite element nodal point coordinates, h_i represents the element interpolation functions, and θ_i are the temperatures at the element nodes. Then the finite element formulation of the present problem becomes [7, p. 184]

$$\mathbf{C} \cdot \dot{\theta} + \mathbf{K} \cdot \theta = \mathbf{Q}, \tag{8}$$

where $\mathbf{K}$ is the conductivity matrix, $\mathbf{C}$ is the heat capacity matrix, and $\mathbf{Q}$ is the nodal point heat flow input matrix. At a specific time when the position of the phase front coincides with that of a nodal point

$$\mathbf{Q} = \rho \cdot L \cdot \frac{\mathrm{d}s(t)}{\mathrm{d}t} \tag{9}$$

at the nodal point and $\mathbf{Q} = \mathbf{0}$ at the other nodal points. When the phase front does not coincide with any nodal point

$$\mathbf{Q} = \mathbf{0}. \tag{10}$$

Since it is obvious from a physical point of view that the conditions on the phase front, equations (5) and (6), will continuously affect the temperature distribution in the medium and that a discontinuity in the slope of the temperature profile is to be expected at the phase front, a new scheme is proposed to implement these facts in the finite element method of solution. Assume for the sake of simplicity that the medium can be modelled by a number of two-node elements as shown in Fig. 1. (It can be shown that exactly the same procedure can be used for elements of variable numbers of nodes.) The method of solution requires that an additional

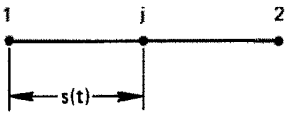

FIG. 1. Typical two-node element.

degree of freedom (i.e. node) be introduced to coincide with the position of the phase front at any instant in time. As the phase front enters the element (c.f. Fig. 1), this node is assumed to separate the original element into two different elements, consistent with the physical effect of the phase front.

The **K** and **C** matrices for the two-node element shown in Fig. 1 prior to the entrance of the phase front are

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}. \tag{11}$$

After the entrance of the phase front into the element in question, new **K** and **C** matrices can be computed for the new elements, by any regular method of integration ([7], p. 155), if we assume that at a certain instant in time, (t), the position of the phase front, $s(t)$, is known:

$$\begin{bmatrix} k'_{11} & k'_{12} & 0 \\ k'_{21} & k'_{22} & k'_{23} \\ 0 & k'_{32} & k'_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c'_{11} & c'_{12} & 0 \\ c'_{21} & c'_{22} & c'_{23} \\ 0 & c'_{32} & c'_{33} \end{bmatrix}. \tag{12}$$

Note that the matrices in equation (12) represent the case when nodes 1 and 2 in Fig. 1 do not belong to the same element and the temperatures in these nodes are not interdependent. The temperature at node j, the position of the phase front, is known to be zero from equation (5). Therefore, in order to solve equation (8) for the temperature distribution at the other nodal points, it is sufficient to introduce in the general **K** and **C** matrices, the following matrix elements from equation (12),

$$\begin{bmatrix} k'_{11} & 0 \\ 0 & k'_{33} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c'_{11} & 0 \\ 0 & c'_{33} \end{bmatrix} \tag{13}$$

in the place of the matrix elements shown in equation (11). Equation (8) can now be solved by any of the usual methods of solution [7, 8]. The stability and convergence will depend on the method chosen. It is obvious that the resulting temperature distribution is consistent with the boundary condition, equation (5). After the nodal temperatures at time (t) are found, equation (8) can be solved on a single element level for node j [see Fig. 1 and equation (12)], to obtain the position of the phase front at the next time step, $s(t + \Delta t)$ by using for the nodal heat flux **Q** at node j,

$$\mathbf{Q}^t = \rho \cdot L \cdot \frac{s(t + \Delta t) - s(t)}{\Delta t}. \tag{14}$$

This expression was obtained by using a simple Euler forward interpolation in the energy balance equation (6). The interpolation used for the solution of equation (8) should be compatible with the one used in equation (14).

The temperature distribution must now be found at time $(t + \Delta t)$ in order to continue the iteration procedure. Since the described scheme is not self-starting, the position of the phase front at the first time step was found by using Neumann's exact solution for a semi-infinite medium [9].

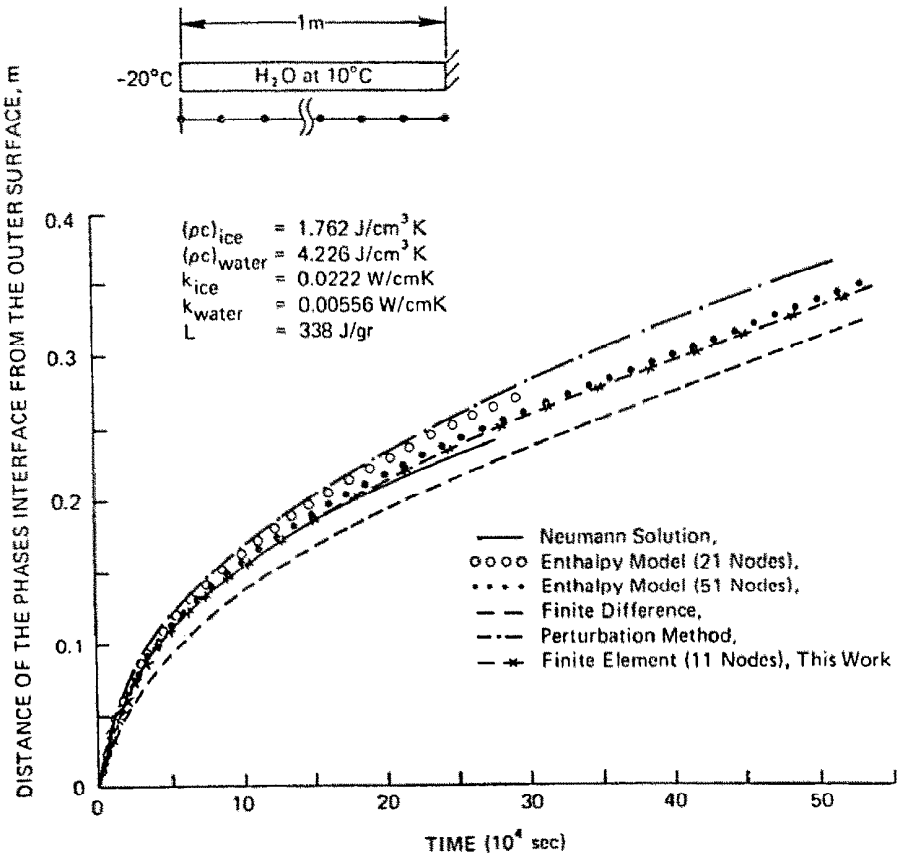

FIG. 2. Position of the phase front as a function of time.

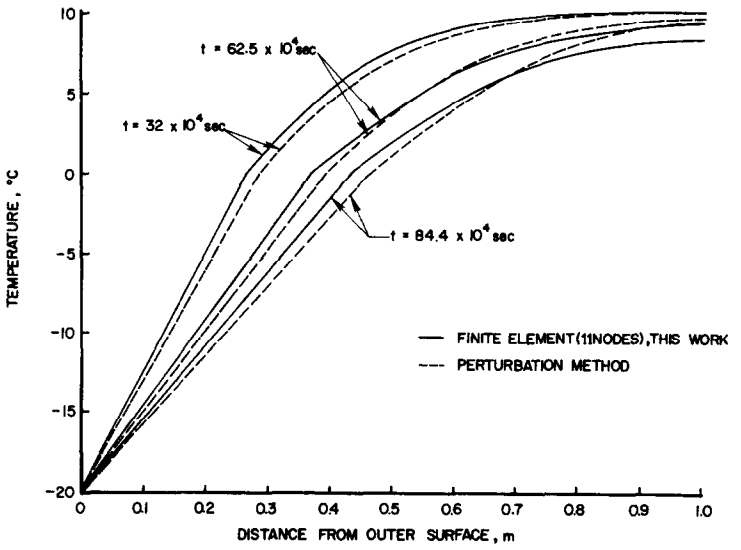

FIG. 3. Temperature distribution as a function of time.

SAMPLE SOLUTION

To illustrate the method, the position of the phase front and the temperature distribution were found for a slab-like region of water with an initial temperature of 10°C when a temperature of − 20°C is applied on the outer surface, Fig. 2. The slab was modelled by using 10 two-node elements with a time step of 4000 s.

In Fig. 2 the position of the phase front obtained here is compared to: an enthalpy model finite element solution using 50 and 20 two-node elements, with a time step of 200 s [6]; Neumann's exact solution for a semi-infinite medium [9]; a finite difference solution, using 20 nodes; and a first-order perturbation solution [10]. Although there is no exact solution for the problem discussed here, it is seen that for short times, in which the Neumann's solution is exact, the method of solution presented here gives an almost identical solution to Neumann's solution. For longer times this solution compared well with the solution found with the 50 two-node element mesh [6]. In Fig. 3 it is seen that the temperature distribution obtained by the finite element method compares well with that obtained by a perturbation method [10].

CONCLUSIONS

A finite element method for the analysis of 1-dimensional change of phase problems has been presented. By this method problems in which the change of phase occurs at a specific temperature can be solved. An example has shown that the results obtained by this method compare well with results found in the literature.

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